ENUMERATION OF DELTAHEDRAL GRAPHS WITH UP TO 10 VERTICES

Naoya TSURUTA, Jun MITANI, Yoshihiro KANAMORI and Yukio FUKUI
University of Tsukuba, Japan

ABSTRACT: In this paper, we enumerate the polyhedral graphs that are realizable as deltahedra with up to ten vertices. We call these "deltahedral graphs". This result was achieved by an experimental approach that trying to construct deltahedra from each of the simple cubic polyhedral graphs. We also provide examples of the graphs that are not realizable as deltahedra. We show that the infinite families of such nonrealizable graphs can be obtained by solving the graph isomorphism problem.

Keywords: Deltahedron, Polyhedral Graph, Geometric Realization

1. INTRODUCTION

A deltahedron is a polyhedron whose faces are congruent equilateral triangles. Only eight of these are convex: those having 4, 6, 8, 10, 12, 14, 16, or 20 faces [5]. Coplanar faces sharing an edge are not allowed. The tetrahedron, octahedron, and icosahedron are the three deltahedra that are regular solids. If we permit nonconvex shapes, then the number of deltahedra is infinite, because we can compose larger deltahedra by attaching two smaller deltahedra.

There are several subclasses of deltahedra. Cundy listed 17 biform deltahedra, which have only two forms of vertices [4]. Olshevsky added another 11 biform deltahedra to Cundy's list [9]. These lists did not permit intersecting faces, so these biform deltahedra are solids. Shephard presented 34 isohedral deltahedra, and McNeill added six further examples to Shephard's list [8, 13]. Isohedral deltahedra are face-transitive and may include intersecting faces. Trigg defined spiral deltahedra as those constructed from strips of equilateral triangles [14].

Each of these classes of deltahedra have their own particular properties. Therefore, the configurations of the vertices are very limited. We decided to loosen the conditions and see what kinds of shapes are possible in the world of deltahedra. In this study, we tried to construct deltahedra from each of the simple polyhedral graphs and counted the graphs that can be constructed as deltahedra, that is, the deltahedral graphs. It is hard to determine whether a graph will form a deltahedron by examining only its structure. Thus, we solve a geometric realization problem, which is the problem of determining whether a triangulation of an orientable surface can be realized geometrically in R^3 as a polyhedron without self-intersections [6].

We present an enumeration of all deltahedral graphs with up to ten vertices and provide examples of the constructed deltahedra. In our realization process, we generate an initial polyhedron with nonequilateral triangles and then deform the faces into equilateral triangles by a gradient method, because the graph does not provide the locations of the vertices. Olshevsky focused on the operations used to construct the deltahedra [9]. Augmentation is an operation that joins each appendage polyhedron to its own single-core face. These simple operations can be detected by solving a graph isomorphism problem. We show that this is useful for finding an infinite family of graphs that are nonrealizable as deltahedra. Note that our result of an enumeration is not theoretical. We provide a realization process that constructs a deltahedron from a graph, but it does not necessarily guarantee the nonrealizability of a graph.

Our deltahedral realization problem is a particular case of a geometric realization problem. In general, Bokowski and Guedes de Oliveira [1] showed that there is a nonrealizable triangulation of the orientable surface of genus 6, and Schewe [12] showed that we can construct nonrealizable triangulations for any number of vertices genus 5 or 6. However, for surfaces of genus $1 \leq g \leq 4$, the problem remains open. The conditions for deltahedral realization are stricter than those. Each face must be realized as an equilateral triangle, and it is necessary to calculate the geometric coordinates to determine whether there exist self-intersections or edges whose dihedral angle is equal to 180°. In this paper, we focus on a surface of genus 0. Previous studies of deltahedra mentioned above also focused on the genus-0 surface. Although a few deltahedra with $g > 0$ are known, we know of no published studies.

2. DELTAHEDRAL GRAPH

Polyhedral graphs are three-connected planar graphs. These graphs contain not only triangular faces, but polygonal faces which have more than four edges. A cubic polyhedral graph is a three-connected cubic planar graph and has only triangular faces. This graph is realized as a polyhedron whose faces are triangles, that is, a simple polyhedron. Deltahedra are a subclass of simple polyhedra because they are composed of equilateral triangles. Therefore, the graphs of deltahedra are a subclass of the graphs of simple polyhedra. The relation between them is shown in Figure 1.

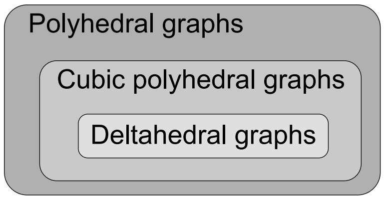


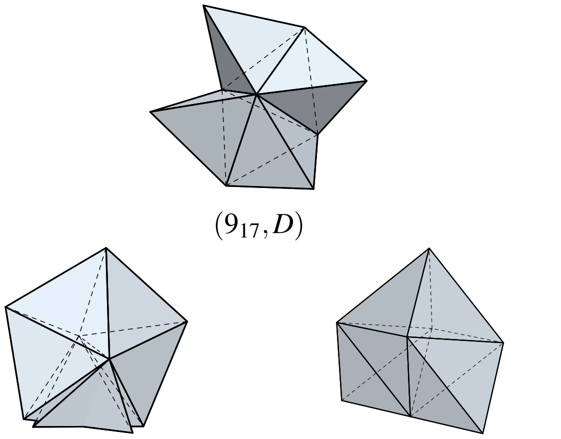
Figure 1: Relation between polyhedral graphs and deltahedral graphs.

Here we define a deltahedral graph as a graph which can be realized as a deltahedron. Although there are various kinds of deltahedra, we will include only deltahedron that do not have any self-intersections and do not have any edges for which the dihedral angle is 180°. For example, all the polyhedra in Figure 2 are composed of congruent equilateral triangles. However, the lower left one $(9_{24}, N)$ has intersecting faces, and the lower right one $(8_{12}, N)$ has coplanar and connecting faces. In this paper, we will not consider deltahedron like these lower ones but only those like the topmost example $(9_{17}, D)$ in Figure 2. The code below each figure is composed of two numbers and a character. The numbers represent the number of vertices and the index of the graph. The index follows the order of an existing graph generation algorithm [3], and they are classified into one of two categories: '*D*' for deltahedral graphs, and '*N*' for nondeltahedral graphs. For example, $(6_1, D)$ is the first six-vertex deltahedral graph that is generated by that algorithm. The important thing is that more than one deltahedra may be obtained from a single graph. Figure 3 shows a deltahedral graph that has one convex form and two nonconvex forms. If a graph has at least one deltahedron, we say it is deltahedral.

3. APPROACH

We enumerate the deltahedral graphs which are combinatorially different. The class of deltahedra is a subset of the class of simple polyhedra whose faces are triangles. Therefore, the number of three-connected cubic polyhedral graphs is an upper bound on the number of deltahedral graphs. Graph enumeration has been widely discussed, and there are many approaches to it. We used the planar graph generation program plantri [3] to obtain the three-connected cubic planar graphs.

We used two steps to realize the graphs. First, each graph was embedded without intersections



$(9_{24}, N)$ has intersecting faces

$(8_{12}, N)$ has coplanar and connecting faces

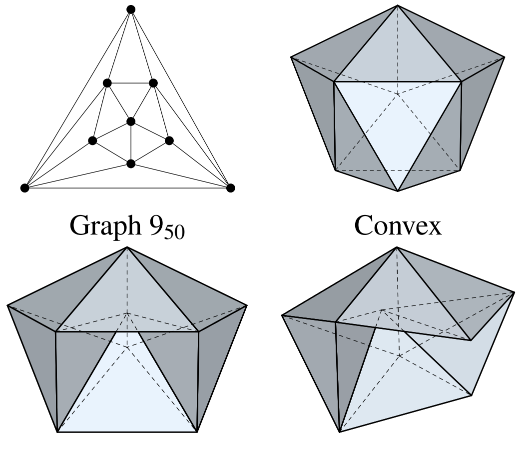
Figure 2: Polyhedra with equilateral triangles. The top polyhedron is a deltahedron and the others are not.

in the 2D plane with straight line edges. Then, we used graph lifting [11] and an iterative deformation process to attempt to construct deltahedra from the graph. A constructed polyhedron may have coplanar neighboring faces or may not even be a solid. The graph was considered to be a deltahedral graph only if the constructed geometry satisfied the conditions for a deltahedron.

As mentioned above, nonconvex deltahedra form an infinite class, and so in order to enumerate them, it is necessary to limit the allowable number of vertices. We chose this limit to be ten. As an upper bound, there are 306 polyhedral graphs which have ten or less vertices. We decided that is enough for the first step of this enumeration problem.

3.1 Graph Embedding

Several methods have been proposed for embedding planar graphs. We used Plestenjak's algorithm, which is based on a spring model [10]. The size of the graph is small enough that it is practical to calculate it. This algorithm chooses a base face and fixes the positions of its vertices in the 2D plane. The remaining vertices are placed inside the base face. We choose the base face randomly and place it so that it forms



Nonconvex Nonconvex

Figure 3: One convex shape and two nonconvex shapes form the graph 9_{50}

an equilateral triangle with edges of unit length. The algorithm calculates the periphericity p_v of each vertex when placing the inner vertices. Periphericity is a kind of centrality and indicates the distance from the outer polygon. The p_v of the outer triangle is 0, and the p_v of the vertices adjacent to them is 1. The p_v increases as going toward the inside. These periphericities are used when generating the initial polyhedron to be used in the iterative deformation. Figure 4(a) is an example of an embedded graph. In this step, the base face is chosen randomly. The results of following steps are different, depending on this initial choice.

3.2 Realization of Deltahedra

First, we generate a polyhedron with nonequilateral triangles from an embedded graph. Although this polyhedron will be deformed to a deltahedron, it should be close to a deltahedron. We obtained the heights h_v for the vertices corresponding to each p_v by using the following formula:

$$h_v = C p_v$$

where C is a constant value that defines the height of the pyramid-like model shown in Figure 4(b). The base face was then shrunk to reduce the differences in the lengths of the edges

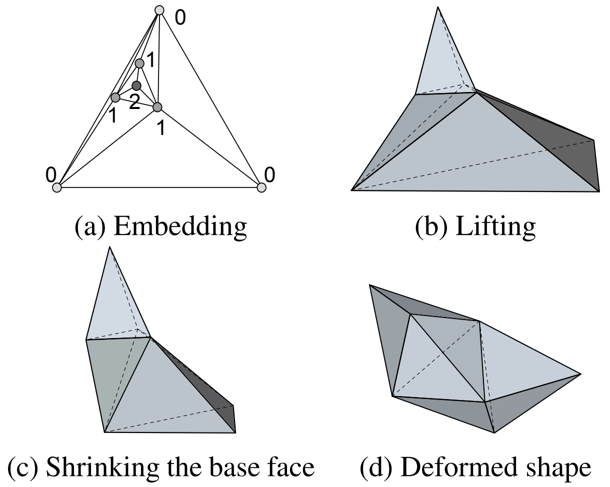


Figure 4: Realization process.

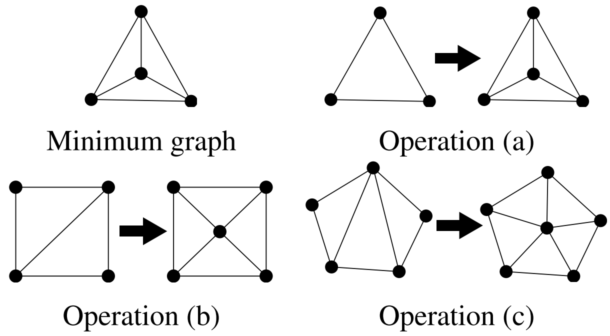


Figure 5: Operations for generation of simple triangulations.

(Figure 4(c)).

Finally, we transformed the generated polyhedron to form a deltahedron by using numerical optimization. We defined a penalty function F which measures the difference in length of the edges, and we then minimized it:

$$F := \sum_i (L(E_i) - 1.0)^2$$

where $L(E_i)$ is the length of E_i. We used the Levenberg-Marquardt algorithm for the iterations and the Gauss-Seidel method for solving the linear equations within each iteration. The resulting polyhedron may include intersecting faces. When this was the case, we tried a different embedding or manually reconfigured the positions of the vertices.

Convergence of this method is not guaranteed. However, there exists a vertex configuration that makes all the lengths of the edges equal when self intersection is not considered. The proof is as follows. There are only three operations for generating all triangulations[2] including:

a) Adding a vertex of degree 3
b) Removing an edge and adding a vertex of degree 4
c) Removing two edges and adding a vertex of degree 5

The minimum four-vertex graph is realized as a tetrahedron. The graph and the operations are shown in Figure 5. When these operations are applied to it, the resultant polyhedron can be fit within an inside region of the tetrahedron because our iterative deformation algorithm does not care about intersecting faces. Operation (a) adds a vertex that separates a face into three faces. The additional faces can be realized as an excavation of a tetrahedron. The additional faces generated by operation (b) will be realized as overlapping coplanar faces resulting in turning over other faces. Operation (c) also causes the turning over of the whole shape, although the shape will still fit within the tetrahedron.

4. RESULTS

Here we present the results of the enumeration and reconstructions. Table 1 shows the numbers of deltahedral and nondeltahedral graphs. We can see that more than half of the graphs are deltahedral graphs. The percentage of nondeltahedral graphs is seen to gradually decrease. To confirm this trend, it may be necessary to investigate larger graphs. Figures 6 and 7 show the constructed polyhedra with seven and eight vertices, respectively. Note that each figure is one of various possible polyhedra. We did not enumerate all the possible realization shapes for each graph. Fortunately, if we do not allow faces to intersect, the variations are small in graphs with ten or fewer vertices. It is easy to manually identify whether a graph has different shapes. For

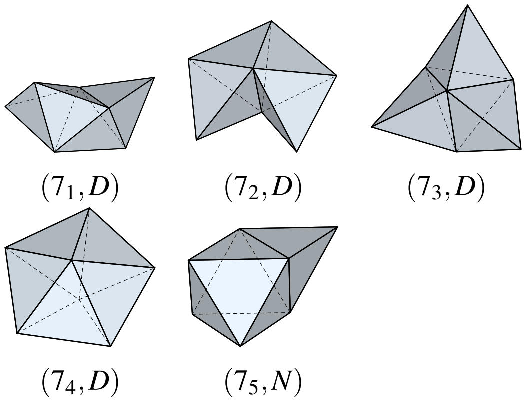


Figure 6: Constructed polyhedra with $V = 7$.

difficult cases, we manually generated *good* initial polyhedra and performed an iterative deformation.

The lengths of all the edges of the constructed polyhedra are very close to 1. Although there is an error caused by the numerical calculations, the maximum difference between the mean edge length and each edge was under 10^{-5}. In most cases, the iteration process converged in a few seconds on a PC with 2.9 GHz Intel Core i7 CPU. The time differed depending on the initial polyhedron produced by the graph lifting.

5. FINDING THE INFINITE FAMILY OF NONDELTAHEDRAL GRAPHS

Some graph structures cause face intersections. For example, in Figure 7, the shape is completely flat in the case of graph 8_7. Similar shapes appear in larger graphs. We can find the graphs that have the paticular nonrealizable structures by comparing graphs.

Figure 8 shows graphs which contain the same partial structures and their realized polyhedra. As shown in Figure 8, when we form a larger deltahedra by connecting two smaller deltahedra along a single face, the original shapes do not change and the graph of the appended deltahedron can be embedded inside a connecting face. This operation can be detected by solving the subgraph isomorphism problem. Hence, we can obtain a family of nondeltahedral graphs from one nondeltahedral graph by solving it, without

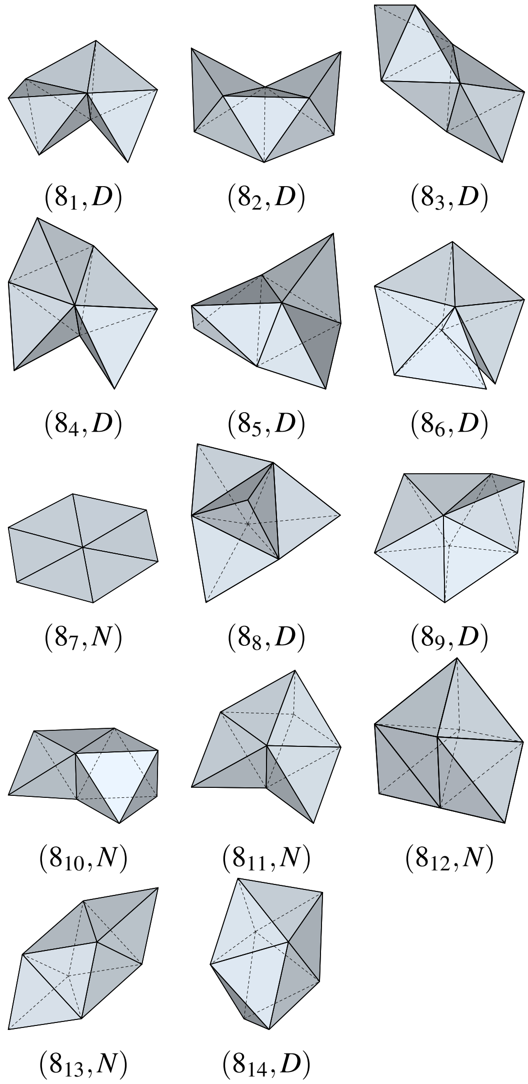


Figure 7: Constructed polyhedra with $V = 8$.

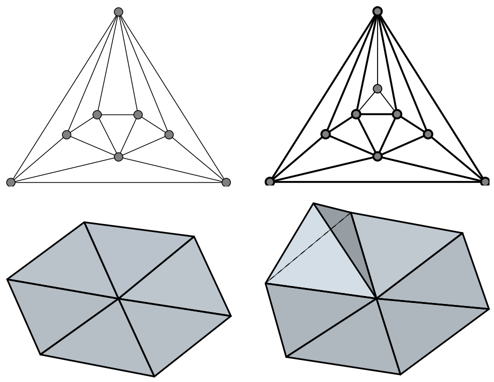
the need to realize the polyhedra.

Figure 9 shows an example of a nondeltahedral family. We used graph 8_7 as a seed and obtained ten nondeltahedral graphs with nine or ten vertices. We used a simple backtracking algorithm for the subgraph isomorphism [15]. The computation times were 140 ms and 2500 ms for nine and ten vertices, respectively.

This subgraph isomorphism only detects the connection of two deltahedra that involves a single face. Figure 10 is a comparison of a connection that involves only one face and with one that involves multiple faces. These shapes look similar, but the isomorphism of the subgraphs only

Table 1: Number of deltahedral graphs.

Vertices	4	5	6	7	8	9	10
Graphs	1	1	2	5	14	50	233
Deltahedral graphs	1	1	2	4	9	36	154
Nondeltahedral graphs	0	0	0	1	5	14	79



$(8_7, N)$ $(9_{26}, N)$

Figure 8: Graphs with and without the appendage tetrahedra, and the associated polyhedra.

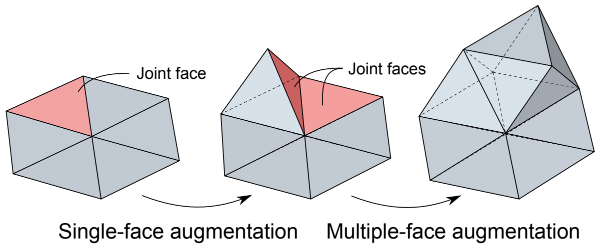


Figure 10: Connection with single faces and with multiple faces.

detects the deformation from left to center. The right one is a seed of an another nondeltahedral family. From center to right, an octahedron is attached along two faces. In this case, the original shapes do not change, but in general, attaching a deltahedron along multiple faces causes deformations in the original shapes.

The graph 7_5 can be realized as a polyhedron that does not contain self-intersections. Such nondeltahedral graphs that are realizable as polyhedra cannot be used as seeds of the subgraph isomorphism problem. A larger graph may be realizable as a deltahedron because the attachment changes the dihedral angles of the edges around the core face.

6. CONCLUSIONS

We have described a method for and the result of an enumeration of deltahedral graphs which have ten or less vertices. Not all simple cubic polyhedral graphs can be realized as deltahedra, due to self-intersections or dihedral angles of 180°. We have also shown that the infinite families of nondeltahedral graphs are obtained by solving the subgraph isomorphism problem. This eliminates the nondeltahedral graphs from the set of cubic polyhedral graphs without the need for realization, and it may be useful for finding the deltahedral ones. Our future work is to improve the discrimination approach by employing this detection method.

The remaining problem is to determine how nonrealizability can be characterized. In order to do this, we need the vertex coordinates because of the dihedral angles. Our deltahedral realization problem is similar to the polyhedral realization problem. We hope that by combining our iterative deformation process with other realization or detection methods [7, 12], we will obtain a method that creates a robust realization of deltahedra.

It is also necessary to investigate graphs which have higher genus numbers. Do all triangulated surfaces with nonzero genus admit a deltahedral realization? Our realization process is not applicable for surfaces with nonzero genus; however, an iterative deformation may be useful. It will be an interesting challenge to find the smallest deltahedron with $g > 0$, such as a toroidal deltahedron.

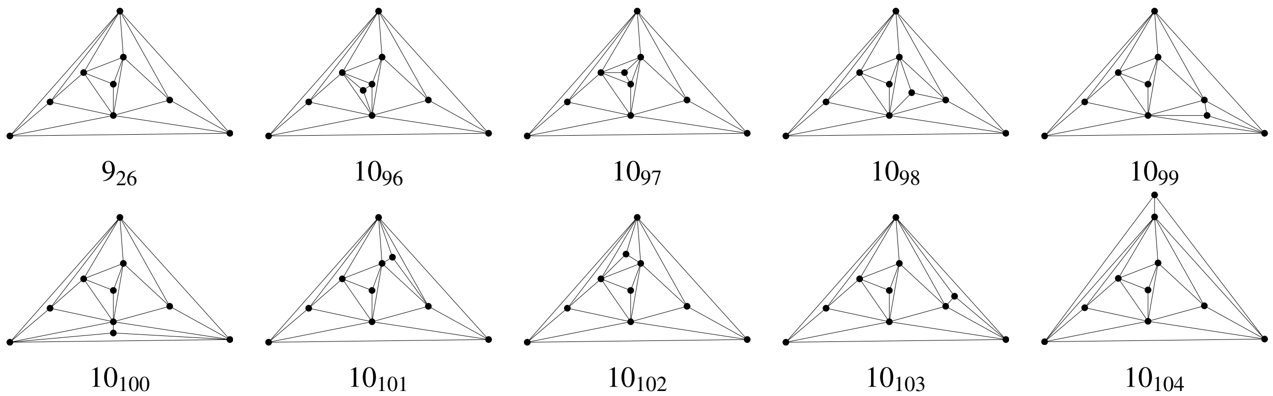


Figure 9: Nondeltahedral family of graph 8_7 up to ten vertices.

REFERENCES

[1] J. Bokowski and A. Guedes de Oliveira. On the generation of oriented matroids. *Discrete & Computational Geometry*, 24(2-3): 197–208, 2000.

[2] R. Bowen and S. Fisk. Generation of triangulations of the sphere. *Mathematics of Computation*, 21(98): pp. 250–252, 1967.

[3] G. Brinkmann and B. D. Mckay. Fast generation of planar graphs. *MATCH Communications in Mathematical and in Computer Chemistry*, 58: 323–357, 2007.

[4] H. M. Cundy. Deltahedra. *Mathmatical Gazette*, 36: 263–266, 1952.

[5] H. Freudenthal and B. L. van der Waerden. On an assertion of euclid. *Simon Stevin*, 25: 115–121, 1947.

[6] B. Grünbaum. *Convex Polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer New York, 2 edition, 2003.

[7] F. H. Lutz. Enumeration and random realization of triangulated surfaces. In A. I. Bobenko, J. M. Sullivan, P. Schröder, and G. M. Ziegler, editors, *Discrete Differential Geometry*, volume 38 of *Oberwolfach Seminars*, pages 235–253. Birkhäuser Basel, 2008.

[8] J. McNeill. Isohedral deltahedra. http://www.orchidpalms.com/polyhedra/rhombic/isohedral_deltahedra.htm.

[9] G. Olshevsky. Breaking cundy's deltahedra record. http://www.interocitors.com/polyhedra/papers/Polytopics 28 - Breaking Cundy's Deltahedra Record.pdf.

[10] B. Plestenjak. An algorithm for drawing planar graphs. *Softw. Pract. Exper.*, 29(11): 973–984, September 1999.

[11] G. Rote. Realizing planar graphs as convex polytopes. In *Proceedings of the 19th International Conference on Graph Drawing*, GD'11, pages 238–241. Springer-Verlag, Berlin, Heidelberg, 2012.

[12] L. Schewe. Nonrealizable minimal vertex triangulations of surfaces: Showing nonrealizability using oriented matroids and satisfiability solvers. *Discrete Comput. Geom.*, 43(2): 289–302, January 2010.

[13] G. Shephard. Isohedral deltahedra. *Periodica Mathematica Hungarica*, 39(1-3): 83–106, 2000.

[14] C. W. Trigg. An infinite class of deltahedra. *Mathematics Magazine*, 51(1): pp. 55–57, 1978.

[15] J. R. Ullmann. An algorithm for subgraph isomorphism. *J. ACM*, 23(1): 31–42, January 1976.

ABOUT THE AUTHORS

1. Naoya Tsuruta, Ph.D. student at Department of Computer Science, University of Tsukuba.

2. Jun Mitani, Associate Professor at Department of Computer Science, University of Tsukuba.

3. Yoshihiro Kanamori, Assistant Professor at Department of Computer Science, University of Tsukuba.

4. Yukio Fukui, Professor at Department of Computer Science, University of Tsukuba.